Tuesday, September 22, 2015

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Problem 19

Problem. Evaluate the definite integral $\int_1^4 \frac{2x+1}{2x}\,dx$.

Solution. This is a rational function where the degree of the numerator is greater than or equal to the degree of the denominator. Therefore, we must use long division. However, this is also a case in which the denominator is a monomial, so we may split the expression up into separate fractions. Either way, we get the same result.

$$\begin{aligned}
\int_1^4 \frac{2x+1}{2x}\,dx &= \int_1^4 \left(\frac{2x}{2x} + \frac{1}{2x}\right)\,dx \\
&= \int_1^4 \left(1 + \frac{1}{2x}\right)\,dx \\
&= \left[x + \frac{1}{2}\ln x\right]_1^4 \\
&= \left(4 + \frac{1}{2}\ln 4\right) - \left(1 + \frac{1}{2}\ln 1\right) \\
&= 3 + \ln 2.
\end{aligned}$$

Problem 47

Problem. Find the indefinite integral $\int xe^{1-x^2}\,dx$.

Solution. Note that the derivative of $1-x^2$ is $-2x$. So let $u = 1-x^2$ and $du = -2x\,dx$. Then

$$\begin{aligned}
\int xe^{1-x^2}\,dx &= -\frac{1}{2}\int (-2x)e^{1-x^2}\,dx \\
&= -\frac{1}{2}\int e^u\,du \\
&= -\frac{1}{2}e^u + C \\
&= -\frac{1}{2}e^{1-x^2} + C.
\end{aligned}$$

Problem 49

Problem. Find the indefinite integral $\int \frac{e^{4x} - e^{2x} + 1}{e^x}\, dx$.

Solution. Just as in problem 19, we have a monomial in the denominator, so we may break the expression up into separate fractions and deal with them individually.

$$\begin{aligned}
\int \frac{e^{4x} - e^{2x} + 1}{e^x}\, dx &= \int \left(\frac{e^{4x}}{e^x} - \frac{e^{2x}}{e^x} + \frac{1}{e^x} \right)\, dx \\
&= \int \left(e^{3x} - e^x + e^{-x} \right)\, dx \\
&= \frac{1}{3}e^{3x} - e^x - e^{-x} + C.
\end{aligned}$$

Problem 66

Problem. Find the indefinite integral $\int \frac{2^{-1/t}}{t^2}\, dt$.

Solution. The obvious substitution to try is $u = -\frac{1}{t}$ and $du = \frac{1}{t^2}\, dt$. This gives us

$$\begin{aligned}
\int \frac{2^{-1/t}}{t^2}\, dt &= \int 2^u\, du \\
&= \frac{2^u}{\ln 2} + C \\
&= \frac{2^{-1/t}}{\ln 2} + C.
\end{aligned}$$

Problem 77

Problem. Find the indefinite integral $\int \frac{1}{e^{2x} + e^{-2x}}\, dx$.

Solution. This could be tricky. Let's try $u = e^{2x}$, $du = 2e^{2x}\, dx$ and see what happens. To obtain the expression $2e^{2x}\, dx$, which is needed for du, let's multiply numerator and denominator by e^{2x} and then adjust by the 2. And note that $e^{4x} = \left(e^{2x}\right)^2$. We get

$$\begin{aligned}\int \frac{1}{e^{2x}+e^{-2x}}\,dx &= \int \frac{e^{2x}}{e^{4x}+1}\,dx\\ &= \frac{1}{2}\int \frac{2e^{2x}}{\left(e^{2x}\right)^2+1}\,dx\\ &= \frac{1}{2}\int \frac{1}{u^2+1}\,du\\ &= \frac{1}{2}\arctan u + C\\ &= \frac{1}{2}\arctan e^{2x} + C.\end{aligned}$$

It worked!

Problem 78

Problem. Find the indefinite integral $\displaystyle\int \frac{1}{3+25x^2}\,dx$.

Solution. Because $25x^2 = (5x)^2$, let's let $u = 5x$ and $du = 5\,dx$. Then

$$\begin{aligned}\int \frac{1}{3+25x^2}\,dx &= \frac{1}{5}\int \frac{5}{3+(5x)^2}\,dx\\ &= \frac{1}{5}\int \frac{1}{3+u^2}\,du.\end{aligned}$$

That's an improvement. Too bad 3 is not a perfect square. Now let $u = \sqrt{3}v$ and $du = \sqrt{3}\,dv$. Then

$$\begin{aligned}\frac{1}{5}\int \frac{1}{3+u^2}\,du &= \frac{1}{5}\int \frac{\sqrt{3}}{3+(\sqrt{3}v)^2}\,dv\\ &= \frac{\sqrt{3}}{5}\int \frac{1}{3+3v^2}\,dv\\ &= \frac{\sqrt{3}}{15}\int \frac{1}{1+v^2}\,dv\\ &= \frac{1}{5\sqrt{3}}\arctan v + C\\ &= \frac{1}{5\sqrt{3}}\arctan \frac{u}{\sqrt{3}} + C\\ &= \frac{1}{5\sqrt{3}}\arctan \frac{5x}{\sqrt{3}} + C\end{aligned}$$

Problem 79

Problem. Find the indefinite integral $\int \frac{x}{\sqrt{1-x^4}}\,dx$.

Solution. Let $u = x^2$ and $du = 2x\,dx$. Then

$$\begin{aligned}\int \frac{x}{\sqrt{1-x^4}}\,dx &= \frac{1}{2}\int \frac{2x}{\sqrt{1-(x^2)^2}}\,dx \\ &= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du \\ &= \frac{1}{2}\arcsin u + C \\ &= \frac{1}{2}\arcsin x^2 + C.\end{aligned}$$

Problem 95

Problem. Find the indefinite integral $\int \frac{1}{9-4x^2}\,dx$.

Solution. Let $u = 2x$ and $du = 2\,dx$. Then

$$\begin{aligned}\int \frac{1}{9-4x^2}\,dx &= \frac{1}{2}\int \frac{2}{9-(2x)^2}\,dx \\ &= \frac{1}{2}\int \frac{1}{9-u^2}\,du.\end{aligned}$$

Now use the general formula or use another substitution. Let $u = 3v$ and $du = 3\,dv$.

Then

$$\begin{aligned}
\frac{1}{2}\int \frac{1}{9-u^2}\,du &= \frac{1}{2}\int \frac{3}{9-(3v)^2}\,dv \\
&= \frac{1}{2}\int \frac{3}{9-9v^2}\,dv \\
&= \frac{1}{6}\int \frac{1}{1-v^2}\,dv \\
&= \frac{1}{6}\cdot 12\ln\left|\frac{1+v}{1-v}\right| + C \\
&= \frac{1}{12}\ln\left|\frac{1+\frac{u}{3}}{1-\frac{u}{3}}\right| + C \\
&= \frac{1}{12}\ln\left|\frac{3+u}{3-u}\right| + C \\
&= \frac{1}{12}\ln\left|\frac{3+2x}{3-2x}\right| + C
\end{aligned}$$

Problem 96

Problem. Find the indefinite integral $\displaystyle\int \frac{x}{\sqrt{x^4-1}}\,dx$.

Solution. This problem looks a lot like problem 79. Let's try the same substitution. Let $u = x^2$ and $du = 2x\,dx$. Then

$$\begin{aligned}
\int \frac{x}{\sqrt{x^4-1}}\,dx &= \frac{1}{2}\int \frac{2x}{\sqrt{(x^2)^2-1}}\,dx \\
&= \frac{1}{2}\int \frac{1}{\sqrt{u^2-1}}\,du \\
&= \frac{1}{2}\ln\left(u+\sqrt{u^2-1}\right) + C \\
&= \frac{1}{2}\ln\left(x^2+\sqrt{x^4-1}\right) + C.
\end{aligned}$$